3D DYNAMICS

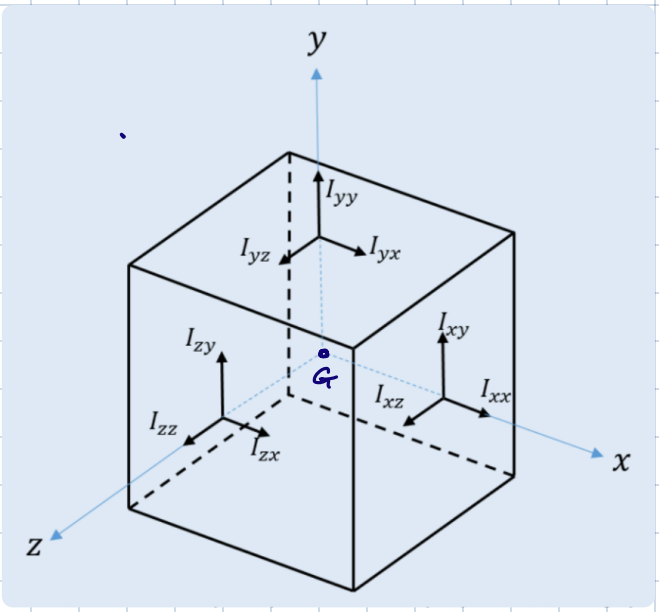
MOMENTS OF INERTIA (MoIs) of a body about an axis measure the distribution of the body's mass about that axis and are always POSITIVE. The smaller the inertia the more concentrated the mass is about the axis.

$$I^G_{xx} = \int (y^2 + z^2)\, dm$$

$$I^G_{yy} = \int (x^2 + z^2)\, dm$$

$$I^G_{zz} = \int (x^2 + y^2)\, dm$$

→ about xyz axis



PARALLEL-AXES THEOREM FOR MoIs - The inertia I^A_{ii} of a body about an axis i passing through a point A is equal to the sum of the inertia I^G_{ii} of the body about a parallel axis through the center of mass G plus the mass M times the square of the shortest distance d_i between the two parallel axes.

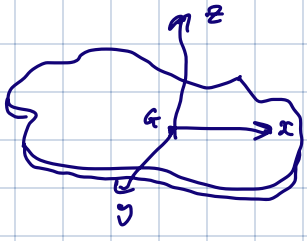
$$I^A_{ii} = I^G_{ii} + M d_i^2 \quad (i = x, y \text{ or } z)$$

* The minimum MoIs of a body occur about axes passing through mass center. All other inertias must be larger calculated by addition of Md^2 term.

PRODUCTS OF INERTIA (PoIs) are taken with respect to planes xy, yz, xz instead of with respect to axes like MoIs. They are indicators of symmetry. Can be POSITIVE, NEGATIVE, ZERO. If a plane is a plane of symmetry, then the products of inertia associated with any axis perpendicular to that plane are zero.

e.g. if xy = plane of symmetry

$$I_{xz} = I_{yz} = 0$$



$$I_{xy} = I_{yx}$$

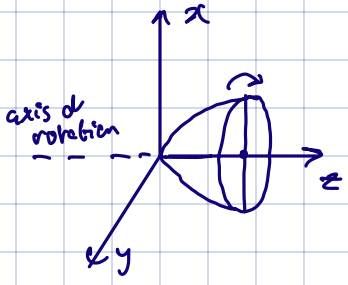
$$I_{xz} = I_{zx}$$

$$I_{yz} = I_{zy}$$

Bodies of revolution have two planes of symmetry.

e.g. if $xz = yz$ = plane of symmetry

$$I_{xy} = I_{yz} = I_{xz} = 0$$



PARALLEL AXIS THEOREM for PoIs - The PoI I^A_{ij} of a body about a pair of axes (i,j) passing through any point A is equal to the sum of the PoI I^G_{ij} of the body about a set of parallel axes through the mass center G plus the mass M times the product of the coordinates c_i and c_j of A relative to G (or G relative to A) measured along these axes.

$$I^A_{ij} = I^G_{ij} + M c_i c_j \quad (i = x, y \text{ or } z \text{ and } j = x, y \text{ or } z)$$

THE INERTIA MATRIX

The MoIs and PoIs of a body about a set of axes passing through some point can be collected into a single inertia matrix.

$$I = \begin{bmatrix} I_{xx} & -I_{xy} & -I_{xz} \\ -I_{yx} & I_{yy} & -I_{yz} \\ -I_{zx} & -I_{zy} & I_{zz} \end{bmatrix}$$

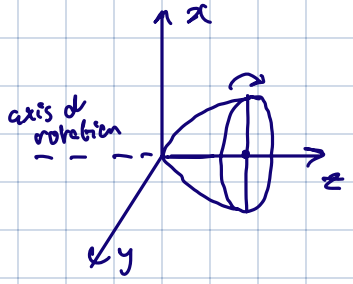
* diagonal entries = MoIs
off diagonal entries = PoIs

For nonsymmetric bodies:

- Can be an infinite number of inertia matrices associated with each point of a body because the inertia matrix changes with orientation of the axes at that point.
- There is generally only one set of axes for each point for which the inertia matrix is diagonal (all PoIs = 0). These axes are called **PRINCIPLE AXES** and the MoIs about these axes are called principle MoIs.
- In general, the principle axes and MoIs are different for each point of a body.

For symmetric bodies:

- Can be multiple sets of principle axes at a given point
- Multiple points can have the same principle axes.
- E.g. x and y axes are principle axes for any point along the z axis and they can be rotated any angle about the z axis to produce another set of principle axes.



All inertia matrices are symmetric → they have real eigenvalues and eigenvectors.

eigenvalues = principle MoIs

eigenvectors = principle directions → If normalised, represent three mutually perpendicular unit vectors in the principle directions.

EULER'S EQUATIONS OF MOTION

$\sum \vec{F} = M\vec{a}$ → 3 Equations

Angular momentum → $\vec{H} = I \times \vec{\omega}$ and

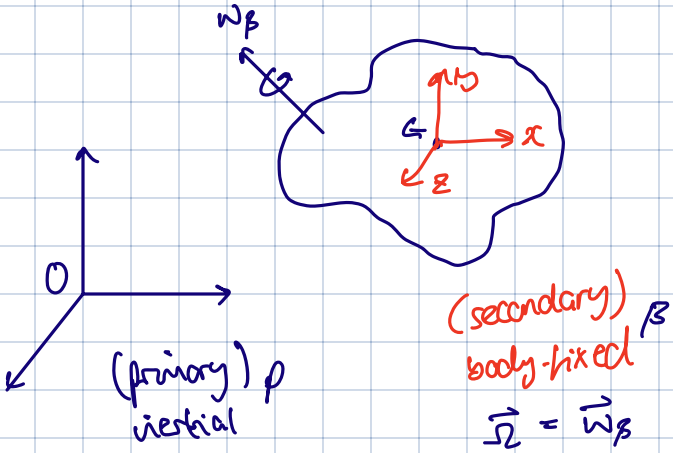
$$\boxed{\sum \vec{M} = \frac{d\vec{H}}{dt} = I\vec{\alpha}}$$

only valid in an inertial frame (no α ∴ non-rotating)

In 3D: limit our point of analysis to a fixed point on body A or mass center G.

$$\sum \vec{M}_A = \frac{d\vec{H}_A}{dt} \quad \text{or} \quad \sum \vec{M}_G = \frac{d\vec{H}_G}{dt}$$

- use a fixed coordinate system to the body so that MOIs and PoIs remain constant.



The secondary coordinate system doesn't need to be at the center of mass G but with the problems given it is a good option as that is normally a set of principle axes MoI's like easier.

$$\left.\frac{d\vec{H}_G}{dt}\right|_\rho = \left.\frac{d\vec{H}_G}{dt}\right|_\beta + \vec{\omega}_\beta \times \vec{H}_G$$

w.r.t primary ← all expressed in secondary xyz →

$$\boxed{\sum \vec{M}_G = I\vec{\alpha}_\beta + \vec{\omega}_\beta \times I\vec{\omega}_\beta}$$

$$\begin{bmatrix} \sum \vec{M}_x \\ \sum \vec{M}_y \\ \sum \vec{M}_z \end{bmatrix} = \begin{bmatrix} I_{xx} & -I_{xy} & -I_{xz} \\ -I_{yx} & I_{yy} & -I_{yz} \\ -I_{zx} & -I_{zy} & I_{zz} \end{bmatrix} \begin{bmatrix} \vec{\alpha}_x \\ \vec{\alpha}_y \\ \vec{\alpha}_z \end{bmatrix} + \begin{bmatrix} 0 & -\vec{\omega}_z & \vec{\omega}_y \\ \vec{\omega}_z & 0 & -\vec{\omega}_x \\ -\vec{\omega}_y & \vec{\omega}_x & 0 \end{bmatrix} \begin{bmatrix} I_{xx} & -I_{xy} & -I_{xz} \\ -I_{yx} & I_{yy} & -I_{yz} \\ -I_{zx} & -I_{zy} & I_{zz} \end{bmatrix} \begin{bmatrix} \vec{\omega}_x \\ \vec{\omega}_y \\ \vec{\omega}_z \end{bmatrix}$$

↓

$$\sum \vec{M}_x = I_{xx}\vec{\alpha}_x + \vec{\omega}_y\vec{\omega}_z(I_{zz} - I_{yy}) + I_{xy}(\vec{\omega}_z\vec{\omega}_x - \vec{\alpha}_y) - I_{xz}(\vec{\omega}_x\vec{\omega}_y + \vec{\alpha}_z) - I_{yz}(\vec{\omega}_y^2 - \vec{\omega}_z^2)$$

$$\sum \vec{M} = I_{yy}\vec{\alpha}_y + \vec{\omega}_x\vec{\omega}_z(I_{xx} - I_{zz}) + I_{yz}(\vec{\omega}_x\vec{\omega}_y - \vec{\alpha}_z) - I_{xy}(\vec{\omega}_z\vec{\omega}_y + \vec{\alpha}_x) - I_{xz}(\vec{\omega}_z^2 - \vec{\omega}_x^2)$$

$$\sum \vec{M} = I_{zz}\vec{\alpha}_z + \vec{\omega}_x\vec{\omega}_y(I_{yy} - I_{xx}) + I_{xz}(\vec{\omega}_y\vec{\omega}_z - \vec{\alpha}_x) - I_{yz}(\vec{\omega}_x\vec{\omega}_z + \vec{\alpha}_y) - I_{xy}(\vec{\omega}_x^2 - \vec{\omega}_y^2)$$

And if we choose the principle frame for analysis, then $I_{xy} = I_{xz} = I_{yz} = 0$

So,

$$\sum \vec{M}_x = I_{xx}\vec{\alpha}_x + \vec{\omega}_y \vec{\omega}_z (I_{zz} - I_{yy})$$

$$\sum \vec{M}_y = I_{yy}\vec{\alpha}_y + \vec{\omega}_x \vec{\omega}_z (I_{xx} - I_{zz})$$

$$\sum \vec{M}_z = I_{zz}\vec{\alpha}_z + \vec{\omega}_x \vec{\omega}_y (I_{yy} - I_{xx})$$

Procedure for solving dynamics problems for rigid bodies in 3D motion.

1) Look at body geometry $\longrightarrow$ look for principle frame.
2) FBD
3) Kinematic relationships
4) EoMs
5) Solve system of equations.