MATHEMATICS INTERNAL ASSESSMENT

Session: May 2025

Analysis and approaches
Level HL

Topic: Mathematical Analysis of Inflatable Tubes Using Torus Equations: Volume, Surface Area, and Weight Capacity.

Number of Pages : 21

(excluding the appendix, work cited and cover sheet)

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Introduction:

"Jump," shouted my brother as I leapt into the deep sea from a ship. He then tossed me a kid-sized inflatable tube, which quickly sank under my weight. This incident sparked my curiosity about the ideal air volume and inflation pressure required to keep a typical person afloat. I observed that most flotation devices resemble a doughnut—a torus in mathematics—which led me to wonder about the optimal float size for a given mass "m" based on its inner radius "R" and ring radius "r." After asking a lifeguard, who explained that a perfect float should not sink more than half its height, I recalled my math class discussions on torus geometry and its applications, and thus began this investigation.

A torus is generated by revolving a circle about a coplanar axis and appears in forms such as ring, horn, and spindle tori. This study examines how internal inflation pressure and torus dimensions affect air volume and buoyant capacity, ultimately guiding the design of effective, safe flotation devices. **This project aims to mathematically model flotation using toroidal volume and surface area relationships, validating these models with real-world measurements.**

Aim:

This investigation explores the relationship between a torus-shaped flotation device's dimensions, internal pressure, and buoyant capacity. Using torus geometry, Archimedes' Principle, and pressure-force equations, it aims to determine the optimal size and inflation pressure needed for effective flotation.

Methodology:

The methodology for this investigation combines rigorous geometric derivations with practical, real-world considerations to optimize the design of a torus-shaped flotation device. First, I derived the volume and partial submersion equations for a torus by integrating over its cross-sectional area, which provided a theoretical basis for understanding how the device's dimensions (minor radius r and major radius R) affect buoyant capacity. I then established a

relationship between internal inflation pressure and supported mass by employing a surface-area method—using half of the total torus surface area—to link pressure, effective area, and buoyant force via P x A= mg. The selected pressure range (0.5 to 3.0 psi) reflects typical operating conditions for consumer-grade inflatable tubes, ensuring that the device is neither too slack nor at risk of rupture. These methods were chosen because they directly relate mathematical theory to practical design.

Variable definition

Variable	Definition	Units
R	**Major radius** of the torus (distance from the centre of the torus hole to the centre of the tube cross-section).	centimetres (cm) or meters (m)
r	**Minor radius** of the torus (radius of the tube's circular cross-section).	centimetres (cm) or meters (m)
x, y	**Cartesian coordinates** used for describing the circle (in 2D) that generates the torus upon revolution.	centimetres (cm) or meters (m)
z	**Vertical coordinate** in 3D space (used along with x, y in parametric/solid of revolution discussions).	centimetres (cm) or meters (m)
θ	**Angular parameter** (minor angle) describing rotation around the smaller (tube) radius.	radians (rad)
ϕ	**Angular parameter** (major angle) describing rotation around the main axis of the torus.	radians (rad)
h	**Submerged height** (often used as 90% of the minor diameter in your scenarios).	centimetres (cm) or meters (m)
V	**Volume** of the torus (or submerged volume) computed via integration or geometric formulas.	cubic centimetres (cm^3) or cubic meters (m^3)
S	**Surface area** of the torus (total or "effective") found via parametric surface equations.	square centimetres (cm^2) or square meters (m^2)
ρ	**Density** of the fluid (usually water) or material, e.g., ρ of water=1000 kg /m^3	kilograms per cubic meter (kg /m^3)
g	**Acceleration due to gravity**, typically 9.81 m /s^2	meters per second squared (m/s^2)
m	**Mass** of the object or person being supported by the flotation device.	kilograms (kg)
F	**Force** (e.g., buoyant force F_b or weight mg).	Newtons (N)
P	**Pressure** inside the inflatable tube (above atmospheric).	pascals (Pa) or pounds per square inch (psi)
A	**Area** (often effective area in contact or contributing to buoyancy).	square centimetres (cm^2) or square meters (m^2)

Background information

Refer to appendix at the end of the IA for explanation of all the mathematical concepts used in explanation, derivation and application.

Mathematical derivations

1.Derivation of the volume of a torus:

Looking at this diagram:

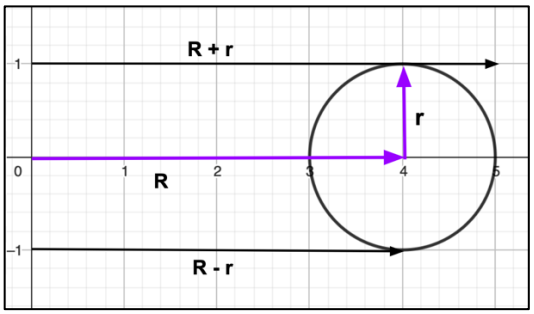


Figure 1: Diagram showing the inner and outer radius

Equation of the circle we will rotate along the y-axis:

$eq1: (x - R)^2 + y^2 = r^2$

This equation can be re-written[1] in terms of y:

$$y = \pm\sqrt{r^2 - (x - R)^2}$$

Now assuming the top and bottom half of the circle:

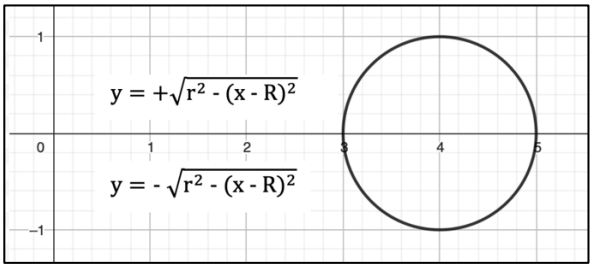


Figure 2: equation of circle displayed

The full equation of the circle is $(x - R)^2 + y^2 = r^2$. The height of the circle at a given x-coordinate is given by:

$$h = 2\sqrt{r^2 - (x - R)^2}$$

[1] Geogebra. "Calculator Suite - GeoGebra." *Www.geogebra.org*, 2001, www.geogebra.org/calculator. Accessed 1 Feb. 2025.

Forming a volume integral out of this would be as shown below using the method of cylindrical shells[2]:

$$V = \pi \int_{b}^{a} distance\ rectangle\ \times height\ of\ the\ rectange\ dx$$

Multiplying by 2 and inserting the limits as mentioned in figure 1:

$$V = 2\pi \int_{R-r}^{R+r} 2x\ \sqrt{r^2 - (x-R)^2}\ dx$$

U-substitution
$x - R = u$
$dx = du$

$$V = 4\pi \int_{R-r}^{R+r} x\ \sqrt{r^2 - (x-R)^2}\ dx$$

$$V = 4\pi \int_{R-r}^{R+r} (u+R)\sqrt{r^2 - (u)^2}\ du$$

$$V = 4\pi \left(\int_{-r}^{r} u\ \sqrt{r^2 - u^2}\ du + \int_{-r}^{r} R\sqrt{r^2 - u^2}\ du \right)$$

$f(x) = u\sqrt{r^2 - u^2}$ is an odd function, so it simplifies to 0.

$$V = 4\pi \left(0 + R \int_{-r}^{r} \sqrt{r^2 - u^2}\ du \right)$$

The part of the equation left is positive, so it will be the upper part of the circle as seen in figure 2, and we know the area of a semi-circle.

Since only this function is left:

$$\int_{-r}^{r} \sqrt{r^2 - u^2}\ du$$

We can observe that this is the area under the curve of the function $\sqrt{r^2 - u^2}$ from -r to r.

Geometrically, this is exactly the area of a semicircle of radius r. The well-known result is:

$$\int_{-r}^{r} \sqrt{r^2 - u^2}\ du = \frac{\pi r^2}{2}$$

$$Geometrically\ the\ area\ of\ a\ semi\ circle\ is\ \frac{\pi r^2}{2}$$

---

[2] Farmer, Jim. "The Volume of a Torus Using Cylindrical and Spherical Coordinates." *Files Eric Ed Gov*, Macquarie University, 22 Jan. 2012, files.eric.ed.gov/fulltext/EJ720058.pdf. Accessed 11 Dec. 2024.

Putting this back in the integral:

$$V = 4\pi\left(R \times \frac{\pi r^2}{2}\right)$$

$$\therefore V = 2\pi^2 r^2 R \qquad Eq\ (1.1)$$

Another way to approximate the flotation volume is by treating the tube as an ellipsoid instead of a torus. The volume of an ellipsoid is given by $V = \frac{4}{3}\pi abc$, where a, b, c are the semi-principal axes. If the inflatable tube deforms into an ellipsoidal shape due to external pressure, this model might provide a more realistic flotation estimate. Moreover, during the derivation of the torus volume, I encountered significant challenges visualizing the rotation of the cross-sectional circle. I had to revisit and refine my trigonometric substitutions multiple times before the results aligned with theoretical expectations. This iterative process not only deepened my understanding of the geometry but also honed my problem-solving skills.

Next, for a torus to be safe in water, it should ideally be more than halfway submerged in water to hold a person floating in the water safely. To find the volume of the torus which already has been submerged, we can find volume in terms of h as seen in figure 3 to help us find the exact height of submersion.

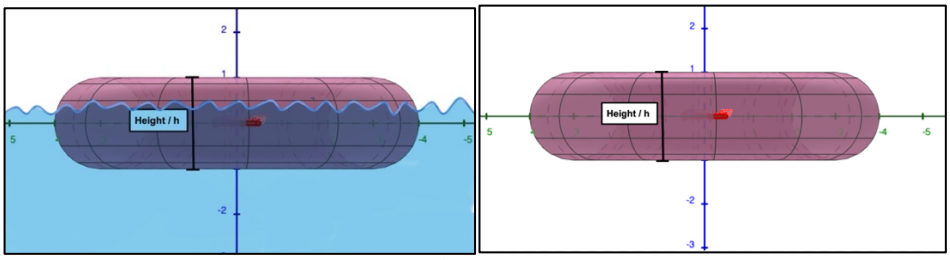


Figure 3: The submerged height showing

2. Deriving volume in terms of height

Using equation 1.1 from above, and assuming A(y) is the area at height y

$$V_{submerged} = \int_{-r}^{-r+h} A(y)\ dy$$

At height y the ring has:

- Outer radius : $R + \sqrt{r^2 - y^2}$
- Inner radius : $R\ -\ \sqrt{r^2 - y^2}$

So the area of the annular cross-section at height y is:

Where A(y) = $\pi\ [(Outer\ radius)^2\ -\ (Inner\ radius)^2]$

Where A(y) = $\pi\ \left[\left(R + \sqrt{r^2 - y^2}\right)^2\ -\ \left(R\ -\ \sqrt{r^2 - y^2}\right)^2\right]$

$$= \pi\left[R^2 + 2R\sqrt{r^2 - y^2} + r^2\ -\ y^2 - \left(R^2 - 2R\sqrt{r^2 - y^2} + r^2\ -\ y^2\right)\right]$$

$$A(y) = \pi(4R\sqrt{r^2 - y^2})$$

Adding this equation back to the volume integral:

$$V_{submerged} = \int_{-r}^{-r+h} 4\pi R\sqrt{r^2 - y^2})\ dy$$

Then using substitution:

Substitution

$y = r\ sin\ \theta$

$dy\ =\ r\ cos\ d\theta$

$$= 4\pi R \int_{\theta_1}^{\theta_2} r\ cos\ \theta \cdot r\ cos\ \theta\ d\theta$$

$$= 4\pi R r^2 \int_{\theta_1}^{\theta_2} (cos\ \theta)^2\ d\theta$$

Using identity

$$(cos\ \theta)^2 = \frac{1 + cos\ 2\theta}{2}$$

$$= 4\pi R r^2 \int_{\theta_1}^{\theta_2} \frac{1 + cos\ 2\theta}{2}\ d\theta$$

$$= 2\pi R r^2 \left[\theta + \frac{sin\ 2\theta}{2}\right]_{\theta_1}^{\theta_2}$$

$$\theta_1 = -\frac{\pi}{2}$$

$$\theta_2 = \frac{-r + h}{r}$$

$$V_{submerged} = 2\pi R r^2 \left[\sin^{-1}\left(\frac{-r + h}{r}\right) + \frac{sin\ 2\left(\sin^{-1}\left(\frac{-r + h}{r}\right)\right)}{2} + \frac{\pi}{2}\right]$$

Simplifying it further:

$$V_{submerged} = 2\pi R r^2 \left[\sin^{-1}\left(\frac{-r + h}{r}\right) + \left(\frac{-r + h}{r}\right) \times \left(\frac{\sqrt{2rh\ -\ h^2}}{r}\right) + \frac{\pi}{2}\right] \qquad eq\ (1.2)$$

While finding this relationship I struggled to visualize how the circular cross-sections translated into a height-based integral, which required repeated checks of the trigonometric relationships and boundary conditions. Wrestling with these details forced me to refine my problem-solving approach, revealing how even small missteps in parameterization could distort the final result. In overcoming these hurdles, I not only strengthened my understanding of the underlying geometry but also gained a deeper appreciation for the elegance of calculus in capturing complex three-dimensional shapes.

3. Finding a relation between r and R

Assume the float is weight is 1kg and a perfect float is one that doesn't submerge more than half of its own stature. I have chosen the height to be 90% of r – since 2r=h, so h=0.9r.

Using eq (1.2) putting h = 0.9r

$$V_{submerged} = 2\pi Rr^2\left[\sin^{-1}\left(\frac{-r+0.9r}{r}\right)+\left(\frac{-r+0.9r}{r}\right)\times\left(\frac{\sqrt{2\cdot r\cdot 0.9r\ -(0.9r)^2}}{r}\right)+\frac{\pi}{2}\right]$$

$$V_{submerged} = 2\pi Rr^2\left[\sin^{-1}(-0.1)+(-0.1)\times\left(\sqrt{0.99}\right)+\frac{\pi}{2}\right]$$

Assuming an average humans mass is 88 kg, we can convert this to volume:

$$88\ kg = 88{,}000\ g = 88{,}000\ cm^3$$

Hence, the volume is $88{,}000\ cm^3$ as calculated above, and making the equation in terms of R we get this.

$$R = \frac{88{,}000}{2\pi r^2\left[\sin^{-1}(-0.1)+(-0.1)\times\left(\sqrt{0.99}\right)+\frac{\pi}{2}\right]} \qquad eq\ (1.3)$$

Graphing this relation looks something like what is below in figure 4

Using this relation we can add a range of values of r to generate the following value of R for accurate torus data. We can use a range between 50-19cm for the major radius keeping in account the average person's waist measurement which approximately has a circumference of

102 cm, with adding a few extra centimetres for the floatation device to be breathable. However, we are taking a limit of 50 cm as a floatation device would be impractical with such large values.

Simplifying the denominator in eq (1.3)

$$\sin^{-1}(-0.1) + (-0.1) \times \left(\sqrt{0.99}\right) + \frac{\pi}{2} = 1.37113$$

$$2\pi \; x \; 1.37113 \approx 8.61$$

Coming up with this equation for a mass supporting 88,000 kg which will be used in scenario 2:

$$R = \frac{88000}{8.61\, r^2}$$



Figure 4: Relationship between R and r for a constant mass of 88 kg

4. Deriving the surface area to find the pressure inside a torus

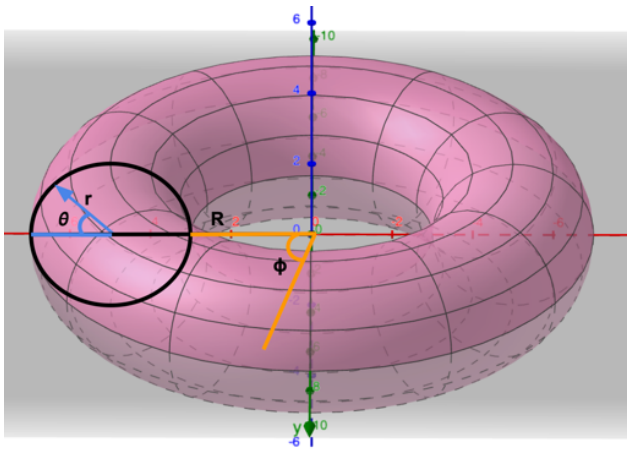


Figure 5: Figure showing variables used in parametric equations

Consider the torus above in terms of parametric equations[3] [4]:

$$x(\theta, \phi) = (R + r\ cos\ \theta)\ cos\ \phi$$

$$y(\theta, \phi) = (R + rcos\theta)\ sin\ \phi$$

$$z(\theta) = r\ sin\ \theta$$

where, as seen in figure 5:
θ : The movement around the minor tube
ϕ : The movement around the major tube
$\overrightarrow{r_\phi}$: The partial derivative with respect to the angle around the major circle as seen in figure 5
$\overrightarrow{r_\theta}$: The partial derivative with respect to the angle around the minor circle as seen in figure 5

Writing the parametric equations in vector form:

$$\overrightarrow{r} = \begin{pmatrix} (R + r\ cos\ \theta)\ cos\ \phi \\ (R + rcos\theta)\ sin\ \phi \\ r\ sin\ \theta \end{pmatrix}$$

$$\overrightarrow{r_\theta} = \begin{pmatrix} -\ sin\ \theta\ cos\ \phi \\ -\ sin\ \theta\ cos\ \phi \\ r\ cos\ \theta \end{pmatrix}$$

$$\overrightarrow{r_\phi} = \begin{pmatrix} -\ (R + r\ cos\ \theta)\ sin\ \phi \\ (R + r\ cos\ \theta)\ cos\ \phi \\ 0 \end{pmatrix}$$

Finding the normal vector of $\overrightarrow{r_\theta}$ and $\overrightarrow{r_\phi}$:

$$\overrightarrow{\alpha}\ :\ \overrightarrow{r_\theta}\ \times\ \overrightarrow{r_\phi} = \begin{pmatrix} -\ r(R + r\ cos\ \theta)\ cos\ \theta\ cos\ \phi \\ -\ r(R + r\ cos\ \theta)\ cos\ \theta\ sin\ \phi \\ -\ r(R + r\ cos\ \theta)\ sin\ \theta \end{pmatrix}$$

Finding the length:

$$|\alpha| =\ r(R + r\ cos\ \theta)\sqrt{(cos\ \theta)^2(cos\ \phi)^2 + (cos\ \theta)^2(sin\ \phi)^2 + (sin\ \theta)^2}$$

Simplifying the square root to 1 using trigonometric identities:

$$|\alpha| = r(R + r\ cos\ \theta)\ \times 1$$

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[3] Vardhan Madahar, Keerti. *Calculus and Geometry*. 2013, www.ias.ac.in/article/fulltext/reso/018/07/0654-0661. Accessed 13 Oct. 2024.
[4] Geogebra. "Calculator Suite - GeoGebra." *Www.geogebra.org*, 2001, www.geogebra.org/calculator. Accessed 1 Feb. 2025.

Finding the area of the this, using double integration to find the surface area of a partial differential equation:

$$\iint_{lower\ limit}^{upper\ limit} r(R + r\ cos\ \theta)\ d\theta\ d\phi$$

We can set the upper limit as 2π and the lower as 0 to solve the integrals :

$$\iint_{0}^{2\pi} r(R + r\ cos\ \theta)\ d\theta\ d\phi$$

$$= r\int_{0}^{2\pi} d\phi \cdot \int_{0}^{2\pi} (R + r\ cos\ \theta)\ d\theta$$

The second integral simplifies to 0 and the first one simplifies to the surface area[5]:

$$= 2\pi r \cdot 2\pi \cdot R$$

$$\therefore Surface\ area = 4\pi^2 rR \qquad eq\ (1.4)$$

5. Deriving a relation between surface area, pressure, and mass.

Now that we have calculated the surface area, using eq (1.4) we can now find the pressure:

Recalling Archimedes principle for buoyancy, when an object is floating the

Buoyant force = weight of object

$$F_b = \rho_{water} \cdot V_{submerged} \cdot g$$

$$\therefore mg = \rho_{water} \cdot V_{submerged} \cdot g$$

Recalling the pressure formula:

$V_{submerged}: 1.8\pi^2 Rr^3$
$Surface\ area: 4\pi^2 rR$
$g: 9.81\ m/s^2$
$\rho_{water}: 1000\ kg/m^3$

$$P = \frac{Force}{Area} \qquad eq(1.5)$$

$$P = \frac{\rho_{water} \cdot V_{submerged} \cdot g}{Area}$$

Inserting the values from above instead of force and area:

[5] Surface. "Surface Area of a Torus." *Mathematics Stack Exchange*, 7 June 2015, math.stackexchange.com/questions/1315349/surface-area-of-a-torus. Accessed 11 Dec. 2024.

$$P = \frac{1000 \cdot 1.8\pi^2 R r^3 \cdot 9.81}{4\pi^2 r R}$$

This can simplify the equation of pressure:

$$P = 4414.5\, r^2 \qquad eq(1.6)$$

Writing this in terms of mass and surface area as well:

$$P = \frac{9.8\ \cdot mass}{Surface\ area} \qquad eq(1.7)$$

Scenario 1: Taking different dimensions of a torus and modelling between

Sample calculation for float A:

Taking floatation device A for example:

$$r = 14$$

$$R = 25$$

$$h = 0.9 \times 14 = 12.6$$

Using eq(1.2) to find the submerged volume:

$$V_{submerged} = 2\pi R r^2 \left[\sin^{-1}\left(\frac{-r + 0.9r}{r}\right) + \left(\frac{-r + 0.9r}{r}\right) \times \left(\frac{\sqrt{2 \cdot r \cdot 0.9r\ - (0.9r)^2}}{r}\right) + \frac{\pi}{2}\right]$$

$$V_{submerged} = 2\pi R r^2 \times 1.37$$

$$= 8.61 \cdot R r^2$$

$$= 8.61 \cdot 25 \cdot 14^2$$

$$= 42189.00\ cm^2$$

Converting this to m^2:

$$\frac{42189}{1000000} = 0.04219\ m^2$$

To find the upthrust, we will use the following equation:

$$Upthrust = volume\ \times\ density\ of\ liquid\ medium\ \times gravity$$

$$= 0.04219 \times 1000 \times 9.81$$

$$= 413.90\ N$$

After finding the upthrust (N) we can find the mass which can be held by the torus.

$$Mass = \frac{Upthrust}{g}$$

$$= \frac{413.90}{9.81}$$

$$= 42.19\ Kg$$

The following table contains all the values of the sample dimensions of different tori and the calculations done in the sample calculation:

Tabular representation

Table 1: Table with different dimensions showing r and R values with mass that can be lifted

Float no.	r (cm)	R (cm)	h (cm)	Volume (cm^3)	Volume (m^3)	Upthrust (N)	Mass (kg)
A	14.0	25.0	12.60	42189.00	0.04219	413.90	42.19
B	14.5	27.0	13.05	48876.82	0.04888	479.78	48.88
C	15.0	30.0	13.50	58,065.00	0.05807	569.73	58.07
D	15.6	32.0	14.04	67,060.06	0.06706	657.99	67.06
E	16.2	34.0	14.58	76,800.91	0.07680	753.71	76.80
F	17.0	36.0	15.30	89,937.60	0.08994	881.50	89.94
G	17.8	38.0	16.02	103,699.04	0.10370	1,017.30	103.70
H	18.5	40.0	16.65	117,964.00	0.11796	158.26	117.96
I	19.2	42.0	17.28	133,495.94	0.13350	1,309.68	133.50
J	19.8	44.0	17.82	148,600.13	0.14860	1,458.94	148.60
K	20.4	45.0	18.36	161,308.56	0.16131	1,583.48	161.31
L	21.0	46.0	18.90	174,697.86	0.17470	1,715.26	174.70
M	21.6	47.0	19.44	186,298.66	0.18630	1,828.46	186.30
N	22.2	48.0	19.98	203,251.97	0.20325	1,993.91	203.25
O	22.8	49.0	20.52	219,965.14	0.21997	2,159.35	219.97
P	23.4	50.0	21.06	236,085.60	0.23609	2,317.71	236.09
Q	24.0	52.0	21.60	257,483.72	0.25748	2,527.95	257.48
R	24.5	53.0	22.05	274,886.36	0.27489	2,697.73	274.89
S	25.0	54.0	22.50	290,385.00	0.29039	2,851.75	290.39
T	25.7	55.0	23.13	324,003.05	0.32400	3,179.24	324.00

Graphical representation

Graphing a relationship between the major radius R and the mass:

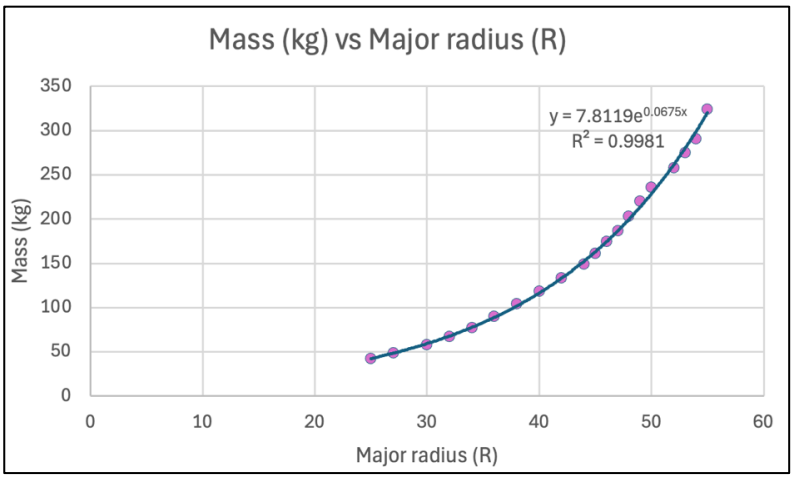


Figure 6: Graph showing relation between mass and major radius with exponential relationship

The data can be graphed using the equation by:

$$m = 7.8119e^{0.0675R}$$

Using this model, we can try and estimate the major radius which can hold 88 kg:

$$88 = 7.8119e^{0.0675R}$$

Solving for x:

$$\frac{88}{7.8119} = e^{0.0675R}$$

Taking ln on both sides:

$$0.0675R = ln\left(\frac{88}{7.8119}\right)$$

$$R = 35.8768\ cm$$

Hence for an approximate weight of 88 kg, the major radius would be 35.88 cm, hence this model can be used for calculating the major radius of a torus. Also, this model is the most accurate compared to others as forming a polynomial model with the equation:

To reflect this in terms of real-world context, we can find the inner radius of the floating torus to compare it with human waists for a proper real-life analysis.

Table 2: Table showing inner radius vs mass

Float no.	R-r (cm) Inner radius	Mass (kg)
A	11	42.19
B	12.5	48.88
C	15	58.07
D	16.4	67.06
E	17.8	76.8
F	19	89.94
G	20.2	103.7
H	21.5	117.96
I	22.8	133.5
J	24.2	148.6
K	24.6	161.31
L	25	174.7
M	25.4	186.3
N	25.8	203.25
O	26.2	219.97
P	26.6	236.09
Q	28	257.48
R	28.5	274.89
S	29	290.39
T	29.3	324

Plotting this data would look like this:

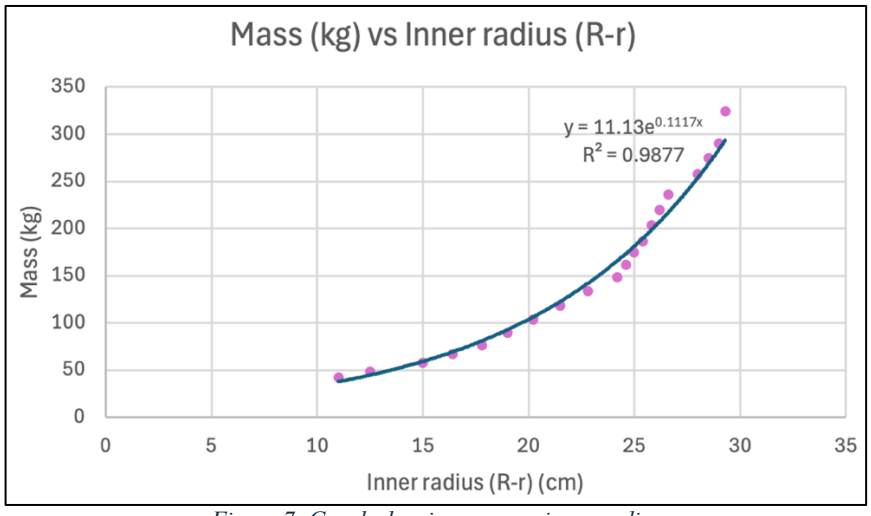


Figure 7: Graph showing mass vs inner radius

Analysis of scenario 1

The following exponential relationship is quite reliable due to the r^2 value being at a quite high value of 0.9877. Moreover, the data can be further validated using real-world websites like https://planetcalc.com/975/ to calculate the upthrust.

The relationship follows an exponential growth model:

$$m = 11.13e^{0.1117R}$$

This depicts as the human radius increases, the required float's mass support grows exponentially. This makes sense because larger body circumferences require a significantly higher volume of displaced water to stay afloat. To compare to real-world data, the average adult waist circumference 75-110 cm which shows that this model can predict which inflatable tube is best suited for each individual.

However, there are some limitations to this model, the assumption that the inflatable tube is 90% submerged may not always hold true as in real-world challenges there may be other factors like water turbulence or currents and other factors which are not accounted for in this model. The model also assumes a cylindrical shape of the body which may not hold true and in real-world scenarios people have different masses and irregular body shapes. Most importantly, the model does not consider variations in body density, assuming all users have the same mass-to-volume ratio. To test the accuracy of this mathematical model, I compared my flotation calculations with real-world manufacturer specifications for inflatable tubes. A typical inflatable ring has a volume of approximately 367 litres (367,000 cm^3) and is rated for a maximum buoyancy of 282 kg. Using my theoretical model, the buoyant force can be estimated by multiplying the volume by the density of freshwater, yielding a predicted flotation capacity of 367 kg. However, in real-world applications, a safety factor is often applied to account for factors such as uneven weight distribution and water disturbances.

Scenario 2: keeping the mass constant and varying the pressure

In this scenario, we analyse the effect of varying air pressure on the flotation capacity of inflatable tubes with fixed dimensions. Unlike Scenario 1, where we determined the relationship between human mass and torus dimensions, here we aim to find the minimum required pressure to keep a fixed mass afloat while ensuring the flotation device remains stable.

We will consider five different flotation tubes from the 20 we analysed earlier. These five were chosen to cover a range of minor and major radii, ensuring a fair comparison across different float sizes. For each tube, we will gradually increase the air pressure from 0.5 psi to 3.0 psi and observe its impact on volume expansion and flotation capacity.

This range of pressure is taken due to:

- **Typical inflatable tubes operate at 0.5 psi – 3.0 psi** (above atmospheric pressure).
- **Below 0.5 psi**, the tube becomes slack, reducing its ability to displace water.
- **Above 3.0 psi**, the material could rupture, making it unsafe.
- A **controlled increase** of 0.5 psi per step allows us to evaluate the optimal inflation level floatation.

Sample calculation for float A:

Using eq(1.4), we can find the area of the tube submerged which we can take as ½ as an estimate for a safe floatation device.

$$Total\ surface\ area = 4\pi^2 Rr$$

Now we can find the total effective surface area (A_{eff}) :

$$A_{eff} = \frac{1}{2} \times 4\pi^2 Rr$$

$$A_{eff} = 2\pi^2 Rr \qquad eq(1.8)$$

This formula assumes that only half the toroidal surface contributes to buoyancy due to water displacement patterns. However, in real-world conditions, this assumption may not always hold, especially for dynamic floating objects.

To ensure surface pressure balance we must follow this condition:

$$P \times A_{eff} = mg$$

Making this in terms of mass:

$$m = \frac{P \times A_{eff}}{g}$$

Using flotation device A:

First calculating the effective area using eq(1.8):

$$A_{eff} = 19.7392 \times 25 \times 14$$

$$= 6908.72\ cm^2$$

Converting to m^2:

$$= 0.6909\ m^2$$

Converting 0.5 psi to Pa:

$$= 0.5 \times 6894.76 = 3447.38\ Pa$$

Calculating force:

$$F = 3447.38\ Pa \times 0.6909\ m^2$$

$$= 2382.4\ N$$

And finally calculating mass:

$$m = \frac{F}{g}$$

$$m = \frac{2382.4}{9.81}$$

$$242.85\ kg$$

Reusing the same floats from scenario 1:

Table 3: Table showing the floats chosen for this scenario

Float	Minor Radius r(cm)	Major Radius R(cm)
A	14.0	25.0
F	17.0	36.0
H	18.5	40.0

Now changing the pressure in these 5 floats:

Table 4: Changing pressure in Float A

P(psi)	Pa	Force(N)	Mass(kg
0.5	3447.38	2382.4	242.85
1.0	6894.76	4764.8	485.7
1.5	10342.14	7147.1	728.1
2.0	13789.52	9529.5	971.4
2.5	17236.90	11911.9	1214.3
3.0	20684.28	14294.3	1457.4

Table 4: Changing the pressure in Float H

P(psi)	Pa	Force(N)	Mass(kg)
0.5	3447.38	5045.7	514.9
1.0	6894.76	10091.4	1029.8
1.5	10342.14	15137.1	1544.7
2.0	13789.52	20182.8	2058.1
2.5	17236.90	25228.5	2573.3
3.0	20684.28	30274.2	=3088.3

Table 5: Changing pressure in Float F

P(psi)	Pa	Force(N)	Mass(kg)
0.5	3447.38	344168.0	425.6
1.0	6894.76	8336.1	850.8
1.5	10342.14	12504.1	1274.6
2.0	13789.52	16672.2	1699.2
2.5	17236.90	20840.2	2124.0
3.0	20684.28	25008.2	2550.5

Graphical representation:

Graphing the following the mass vs pressure graph for all the three floatation devices:

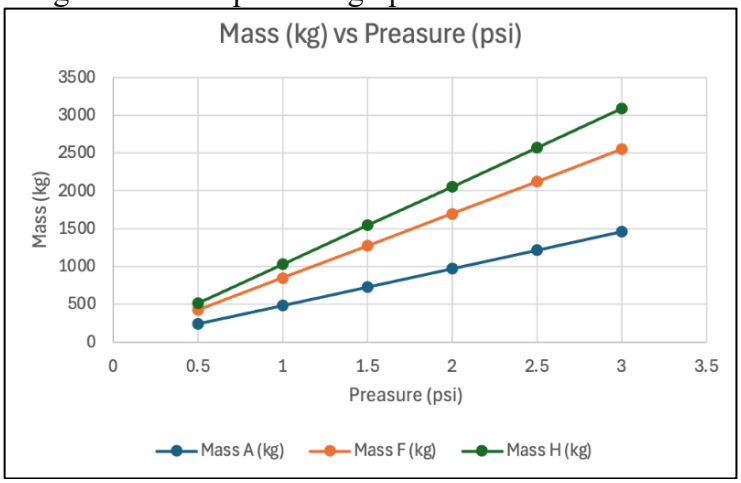


Figure 8: Graph showing mass vs pressure for 3 different dimensions of floats

Analysis

In Scenario 2, the model is refined by linking the internal inflation pressure to the buoyant capacity via the effective surface area of the torus. In this approach, we assume that one-half of the total torus surface area contributes effectively to buoyancy. Given that the full surface area of a torus is $4\pi^2 Rr$, the effective area is defined as $A_{eff} = 2\pi^2 Rr$. Using the pressure balance equation. $P \times A_{eff} = mg$. We derive that the supported mass is given by:

$m = \frac{P \times A_{eff}}{g}$. The internal pressure ,effective area and supported mass have a linear

dependency, which is demonstrated in this relationship. As an example, Float A (with baseline dimensions r equating to 14 cm, and R equal to 25 cm) has an effective area of around 6909 cm squared or 0.6909 square meters. The upward force at inflation pressure of 0.5 psi, which is approximately 3447 Pa, is equal to 3447 times 0.6909 which is approximately 242.85 kg. This mass is far greater than the target of 88 kg. The buoyant force assumption is that half of the surface area of the torus fully contributes to its generation, which explains the high value. In contrast, Scenario 1 employed a partial submersion volume approach which yielded more conservative mass capacities.

Conclusion

The aim of this study was to establish the most effective design specifications for inflatable floating devices, in this case, torus-shaped tubes, by analysing the proportional relationship between their geometric features, internal inflation pressure, and buoyant capacity. One limitation of this study is that inflatable tubes are often deformed due to external pressure and internal air compression. This deviation from the ideal torus shape may affect the actual flotation behaviour.

At the second stage of the investigation (Scenario 2), a different approach was taken which looked at the effect of reducing the geometric dimensions while changing the internal pressure with the goal of increasing buoyant capacity. I was able to show that $P \times A = mg$ defines the mass which can be computed to be supported under the pressure range of 0.5 to 3.0 psi. The results indicated that with this model, higher internal pressures provided significantly larger buoyant force enabling the device to support heavier loads. Comparing the two scenarios highlights the importance of both geometry and inflation pressure in determining a float's performance.

While this IA uses analytical equations to derive flotation characteristics, an alternative approach is to simulate buoyancy numerically using finite element analysis (FEA). This

computational method divides the inflatable tube into small sections and applies fluid dynamics equations to determine submerged volume dynamically.

Further models could be developed by adding dynamic effects of water, material's plasticity, and turbulence of water. Regardless, this particular research makes a reasonable contribution toward the development of inflatable flotation devices by providing suggestions that are realistic and compatible with commercial use. Reflecting on this investigation, I realized that the challenges of reconciling abstract mathematical models with real-world parameters were instrumental in my development as a problem-solver. The difficulties I encountered—particularly in managing complex integrals and ensuring precise variable definitions—highlighted the importance of meticulous attention to detail. These experiences have not only enriched my mathematical understanding but have also underscored the value of perseverance and continuous refinement in research." Emphasize both the significance of your findings and how these lessons might inform future investigations.

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Appendix/ Background information:

These are the mathematical concepts used in the following investigation of a torus:

Appendix 1.Torus geometry

In geometry, a torus (pl.: tori or toruses) is a surface of revolution generated by revolving a circle in three-dimensional space one full revolution about an axis that is coplanar with the circle. The main types of toruses include ring toruses, horn toruses, and spindle toruses.[6] A ring torus is sometimes colloquially referred to as a donut or doughnut.

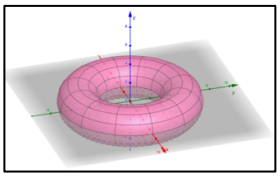


Figure 9: A torus on a 3D diagram and its equation

A torus can be written in 3D terms as:

$$= \begin{pmatrix} (R + r\ cos\ \theta)\ cos\phi \\ (R + r\ cos\ \theta)\ sin\phi \\ r\ sin\ \theta \end{pmatrix}$$

Where u describes the "tube" cross-section and v the revolution around the centre.[7]

A torus can be described using the following equation, where 3.9 is R and 1.4 is r.

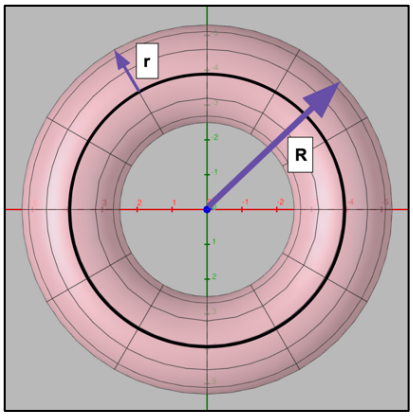


Figure 10: Visual representation of r and R

[6] Pierce, Rod. "Torus." *Mathsisfun.com*, 2017, www.mathsisfun.com/geometry/torus.html. Accessed 3 Jan. 2025.

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Appendix 2. Archimedes principle

It states that "The upward buoyant force that is exerted on a body immersed in a fluid, whether partially or fully submerged, is equal to the weight of the fluid that the body displaces and acts in the upward direction at the centre of mass of the displaced fluid".

$$F_{buoyant} = \rho_{fluid} \cdot g \cdot V_{displaced}$$

Appendix 3.Partial submersion and submerged volume

Assuming the torus is 90% submerged (or any fraction), we calculate only that fraction's volume to find how much fluid is displaced and find the how much weight can be taken by the floatation device.

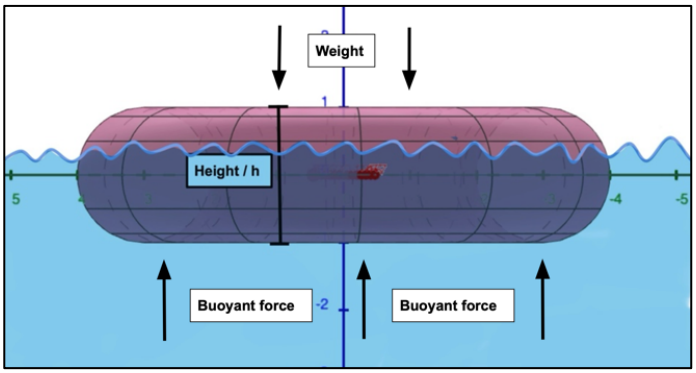


Figure 11: All forces acting on a floatation device, Archimedes' principle

Appendix 4. Internal pressure and surface area

Pressure can be defined as:

$$P = \frac{Force}{Area}$$

Force is measured in Newtons; area in square meters; thus pressure is in pascals.

Appendix 5. Double integration

Double integral is mainly used to find the surface area of a 2d figure, and it is denoted using ∬. We can easily find the area of a rectangular region by double integration. Using the cross product this method helps sum up all the differential area to give the full area of a torus.